Elementary Operations

An elementary operation on a Matrix A over a field F is an operation of the following 3 types —

1. Interchange of two rows (or columns) of A
2. Multiplication of a row (or column) by a non-zero scalar c in F
3. Addition of a scaler multiple of one row (or column) to another row (or column)

Examples:

① $\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow[\text{Interchange of 2nd and 3rd Row}]{R_{23}} \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 7 \\ 4 & 9 & 5 \end{pmatrix}$

② $\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$

$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow[\text{Interchange of 2nd and 3rd Row}]{C_{23}} \begin{pmatrix} 2 & 0 & 4 \\ 4 & 5 & 9 \\ 1 & 7 & 3 \end{pmatrix}$

③ $\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{pmatrix} \xrightarrow[\text{new } R_2]{R_2 - 2R_1 = R_2'} \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{array}{c} R_1 \\ R_2' \\ R_3 \end{array}$

$\begin{pmatrix} 4 \\ 9 \end{pmatrix} \begin{array}{l} \to 4 - 2.2 = 4 - 4 = \\ \to 9 - 2.4 = 9 - 8 = \end{array} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Only the changed row or ~~column~~ column will be changed Other rows or columns will remain unchanged.

④ $\begin{array}{cc} C_1 & C_2 \end{array}$ $\qquad\qquad$ $\begin{array}{cc} C_1 & C_2' \end{array} \to \text{New } C_2$

$\begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{pmatrix} \xrightarrow{C_2 - 2C_1 = C_2'} \begin{pmatrix} 2 & 0 \\ 4 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix} \begin{array}{l} \to 4 - 2.2 = \\ \to 9 - 4.2 = \\ \to 3 - 2.1 = \end{array} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

- **Row-equivalent / Column equivalent:**

A matrix B in S is said to be row-equivalent (column equivalent) to a matrix A in S if B can be obtained by successive application of a finite no. of elementary row operations (Column Operation) on A

- **Row-Reduced:** An $m \times n$ matrix A is called row-reduce if
  (a) the 1st non-zero element in each non-zero row is 1 (leading 1)
  (b) in each column containing the leading 1 of some

row, the leading 1 is the only non zero element.

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

- Row reduced echelon Matrix:

(i) A is row-reduced.

(ii) there is an integer r $(0 \le r \le m)$ such that the 1st r rows of A are non zero rows the remaining rows are all zero rows

(iii) if the leading element of the ith non-zero row occurs in the k_i th column of A, then $k_1 < k_2 < \cdots < k_r$

Row reduced echelon matrix are — $\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

① Obtain a Row reduced matrix which is row equivalent to —
(only row opera.)

→ $\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{pmatrix}$

(To make 1st entry of R_1 is 1 →)

$\big\downarrow \frac{1}{2}R_1$

$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 1 & 3 & 1 & 3 & 1 \end{pmatrix}$

$R_2' = R_2 - 2R_1$ = new R_2 (others rows will remain unchanged)

$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 3 & 1 & 3 & 1 \end{pmatrix}$

$\big\downarrow R_3' = R_3 - R_1$

$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 3 & 0 & 2 & 1 \end{pmatrix}$

$\big\downarrow R_3'' = R_3' - R_2$

$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

no further operator

this is the row-reduced matrix

5

Matrix

(1) Apply elementary row operations to reduce the following matrix to a row echelon matrix

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 6 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 12 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{R_2-3R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{R_3-5R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{R_4-3R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_{34}} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3/2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

no further change

This is the required Matrix

(2) Obtain a row echelon matrix which is row equivalent to —

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

To make 1st element of 1st row is 1

$$\xrightarrow{R_{12}} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 1 & 3 & 1 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 3 & 1 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

$$\xrightarrow[R_4-3R_1]{R_3-R_1} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix} \xrightarrow{-R_3} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix} \xrightarrow{R_1-2R_2} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

7

$$\xrightarrow{R_4+5R_2}\begin{pmatrix}1&3&0&2&1\\0&0&1&1&0\\0&0&0&0&0\\0&0&0&3&3\end{pmatrix}\xrightarrow{R_4/3}\begin{pmatrix}1&3&0&2&1\\0&0&1&1&0\\0&0&0&0&0\\0&0&0&1&1\end{pmatrix}$$

$$\xrightarrow[R_2-2R_4]{R_1-2R_4}\begin{pmatrix}1&3&0&0&-1\\0&0&1&0&-1\\0&0&0&0&0\\0&0&0&1&1\end{pmatrix}\xrightarrow{R_{34}}\begin{pmatrix}1&3&0&0&-1\\0&0&1&0&-1\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}$$

→ Required Matrix

- Note: (1) If a row-reduced echelon matrix R has r non-zero rows, then rank of R is "r".

(2) If a matrix A be row equivalent to a row echelon matrix having 'r' non-zero rows, then rank $A = r$

(3) The rank of a matrix remains invarient under an element- any row operation

(4) An $n\times n$ matrix A is non-singular if A is row-equivalent to the Identity matrix I_n

~~5 Determine rank~~

(4) Determine Rank of $A = \begin{pmatrix}1&2&1&0\\2&4&8&6\\3&6&6&3\end{pmatrix}$

$$\rightarrow \begin{pmatrix}1&2&1&0\\2&4&8&6\\3&6&6&3\end{pmatrix}\xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_2}\begin{pmatrix}1&2&1&0\\1&2&4&3\\1&2&2&1\end{pmatrix}\xrightarrow[R_3-R_1]{R_2-R_1}\begin{pmatrix}1&2&1&0\\0&0&3&3\\0&0&1&1\end{pmatrix}$$

$$\xrightarrow{R_2/3}\begin{pmatrix}1&2&1&0\\0&0&1&1\\0&0&1&1\end{pmatrix}\xrightarrow{R_3-R_2}\begin{pmatrix}1&2&1&0\\0&0&1&1\\0&0&0&0\end{pmatrix}\xrightarrow{R_1-R_2}\begin{pmatrix}1&2&0&-1\\0&0&1&1\\0&0&0&0\end{pmatrix}$$

Here the row-reduced echelon matrix R has 2 non-zero rows.

$\therefore$ Rank of $R = 2$

A is row equivalent to R $\therefore$ Rank of $A = 2$

Matrix

- **Equivalence**: An $m\times n$ matrix B is said to be equivalent to an $m\times n$ matrix A over $\mathbb{F}$, if B can be obtained from A by a finite no. of elementary row and column operations

Row reduced + column reduced

□ Obtain the fully reduced normal form of the matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}$$

$$\xrightarrow{R_3-2R_1} \begin{pmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix} \xrightarrow{R_4-3R_1} \begin{pmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_3-2R_2} \begin{pmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_4-R_2} \begin{pmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_2-3C_1} \begin{pmatrix} 1 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+2R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_5-2C_1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_4-2C_5} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_3-C_5} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>The.</u> A Matrix is a non-singular iff it can be expressed as the product of a finite no. of elementary matrices.

<u>The.</u> An $m\times n$ matrix B is equivalent to an $m\times n$ matrix A iff $B = PAQ$ where P, Q are non-singular matrices.

Matrix

Ex. Find the Inverse of A where $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{pmatrix}$

→ Always set Identity matrix with A like $(A \vdots I_3)$ and make a 3×6 matrix and the perform elementary row operations to a row-reduced echelon matrix.

make 0 corr. to 1^{st} 1 of 1st row

$$\underline{(A \mid I_3)} = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2} R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -\frac{1}{2} & -2 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$I_3 \quad A^{-1}$

$$\therefore A^{-1} = \begin{pmatrix} 8 & -\frac{1}{2} & -2 \\ -1 & \frac{1}{2} & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

Ex. Show that the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & 3 \end{pmatrix}$ is non-singular and expressed it as a product of elementary matrices.

$$\rightarrow \begin{pmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & 3 \end{pmatrix} \xrightarrow[\frac{1}{3} R_2]{\frac{1}{2} R_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 \\ 6 & 2 & 3 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 6 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3 - 6R_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - \frac{1}{2} R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 + \frac{1}{2} R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

A is equivalent to I_3

$\Rightarrow A$ is non-singular as $|I_3| \neq 0 \neq |A|$

write the operations from back

$$\left(R_2 + \tfrac{1}{2} R_3\right)\left(R_1 - \tfrac{1}{2} R_3\right)\left(R_3 - 2R_2\right)\left(R_3 - 6R_1\right)\left(R_2 - R_1\right)\left(\tfrac{1}{3} R_2\right)\left(\tfrac{1}{2} R_1\right) A = I_3$$

$E_{ij}(a) = R_i + aR_j$

or, $E_{23}\left(\frac{1}{2}\right) E_{13}\left(-\frac{1}{2}\right) E_{32}(-2) E_{31}(-6) E_{21}(-1) E_2\left(\frac{1}{3}\right) E_1\left(\frac{1}{2}\right) A = I_3$

Take inverse on bothside:

$\Rightarrow A = \left\{E_1\left(\frac{1}{2}\right)\right\}^{-1} \left\{E_2\left(\frac{1}{3}\right)\right\}^{-1} \left\{E_{21}(-1)\right\}^{-1} \left\{E_{31}(-6)\right\}^{-1} \left\{E_{32}(-2)\right\}^{-1} \left\{E_{13}\left(-\frac{1}{2}\right)\right\}^{-1}$

Matrix

$\{E_{23}(\tfrac{1}{2})\}^{-1} I$

This factorisation is not unique

$= E_1(2)\, E_2(3)\, E_{21}(1)\, E_{31}(6)\, E_{32}(2)\, E_{13}(\tfrac{1}{2})\, E_{23}(-\tfrac{1}{2})$

- Congruent Definition $\Rightarrow$ Theorem 3.8.1

Normal form under congruence $\Rightarrow$ P-107

Rank, Index and Signature $\Rightarrow$ P-107 elementary column operation on A. Such an operation given by the product EAE^t is called a *congruence operation* on A.

Definition.

Let S be the set of all $n \times n$ matrices over a field F. A matrix $B \in S$ is said to be *congruent* to a matrix $A \in S$ if there exists a non-singular matrix $P \in S$ such that $B = P^t AP$.

Since P is non-singular, it can be expressed as the product $E_1E_2 \ldots E_k$ where E's are elementary matrices of order n.

Therefore $B = {E_k}^t {E_{k-1}}^t \ldots {E_2}^t {E_1}^t AE_1E_2 \ldots E_{k-1}E_k$.

Since ${E_i}^t AE_i$ gives a congruence operation on A, B is obtained by applying k successive congruence operations on A.

Since the rank of a matrix remains unaltered by an elementary operation, the rank remains invariant under a congruence operation. In other words, two congruent matrices have the same rank.

Theorem 3.8.1. The relation of congruence on the set of all $n \times n$ matrices over a field F is an equivalence relation.

Proof. Let S be the set of all $n \times n$ matrices over a field F.

Let ρ be the relation of congruence on S.

(i) $A \rho A$ holds for all $A \in S$, because I_n is a non-singular matrix in S and $A = {I_n}^t AI_n$. Therefore ρ is reflexive.

(ii) Let $A, B \in S$ and $A \rho B$ hold. Then there exists a non-singular matrix P in S such that $B = P^t AP$.

$$\begin{aligned} \text{Therefore } A &= (P^t)^{-1} BP^{-1} \\ &= (P^{-1})^t BP^{-1} \\ &= Q^t BQ, \text{ where } Q \in S \text{ and is non-singular.} \end{aligned}$$

This implies $B \rho A$. Thus $A \rho B \Rightarrow B \rho A$ and therefore ρ is symmetric.

(iii) Let $A, B, C \in S$ and $A \rho B$ and $B \rho C$ hold.

Then there exist non-singular matrices P, Q in S such that $B = P^t AP$ and $C = Q^t BQ$.

$$\begin{aligned} \text{Therefore } C &= Q^t (P^t AP) Q \\ &= (PQ)^t A(PQ) \\ &= M^t AM, \text{ where } M \in S \text{ and is non-singular.} \end{aligned}$$

This implies $A \rho C$.

Thus $A \rho B$ and $B \rho C \Rightarrow A \rho C$ and therefore ρ is transitive.

Hence ρ is an equivalence relation on S.

If $M \neq O$, the same procedure as described in the previous cases can be applied to reduce M to a congruent symmetric matrix with a non-zero element in the $(1,1)$ position and zeros elsewhere in the first row and first column of M. All these operations however are to be applied on the matrix C but these will not affect the first row and the first column of C.

Thus A is transformed to a congruent symmetric matrix $G = \begin{pmatrix} g_{11} & & \\ & g_{22} & \\ & & H \end{pmatrix}$, where $g_{11} \neq 0, g_{22} \neq 0, g_{1j} = g_{j1} = 0, j = 2,3,\ldots,n, g_{2j} = g_{j2} = 0, j = 1,3,\ldots,n$ and H is a symmetric matrix of order $(n-2)$. Repeating the same procedure, if necessary, we obtain in a finite number of steps a congruent diagonal matrix $D = (d_{ij})$.

Since the rank remains invariant under congruence operations, rank of $D = r$ and therefore
$$\begin{aligned} d_{ii} &\neq 0, i = 1,2,\ldots,r \\ d_{ii} &= 0, i = r+1, r+2,\ldots,n. \end{aligned}$$

Theorem 3.8.3. An $n \times n$ real symmetric matrix A of rank r is congruent to the diagonal matrix G whose first m diagonal elements are 1, the next $r-m$ diagonal elements are -1 and the remaining diagonal elements, if there be any, are all zero.

Proof. By the previous theorem, A is congruent to the diagonal matrix $D = (d_{ij})$ where $d_{ii} \neq 0, i = 1,2,\ldots,r$ and $d_{ii} = 0, i = r+1, r+2,\ldots,n$.

If the non-zero elements in D be not all positive, without loss of generality, we can assume that first m diagonal elements are positive and the next $r-m$ diagonal elements are negative. Because, if a positive diagonal element appears after a negative element, they can be interchanged by suitable congruence operations.

Next let us apply the row operations $\frac{1}{\sqrt{d_{11}}}R_1, \frac{1}{\sqrt{d_{22}}}R_2, \cdots, \frac{1}{\sqrt{d_{mm}}}R_m, \frac{1}{\sqrt{-d_{m+1m+1}}}R_{m+1}, \cdots, \frac{1}{\sqrt{-d_{rr}}}R_r$ and the corresponding column operations on D.

Then D is transformed to a congruent diagonal matrix $G = (g_{ij})$, where $g_{11} = g_{22} = \cdots = g_{mm} = 1, g_{m+1,m+1} = \cdots = g_{rr} = -1, g_{ii} = 0$, for $i = r+1, r+2,\ldots,n$.

Thus G takes the form $\begin{pmatrix} I_m & & \\ & -I_{r-m} & \\ & & O \end{pmatrix}$.

G is said to be the *normal form of A under congruence.*