4.10. System of Linear Equations.

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \quad \cdots \quad \text{(i)} \end{aligned}$$

where a_{ij}'s and b_i's are elements of a field F, called the field of scalars. a_{ij}'s are called coefficients of the system. In particular, a_{ij}'s and b_i's are real (or complex) numbers when F is the field $\mathbb{R}$ (or $\mathbb{C}$).

An ordered set $(c_1, c_2, \ldots, c_n)$ where $c_i \in F$, is said to be a *solution* of the system (i) if each equation of the system is satisfied by

$$x_1 = c_1, x_2 = c_2, \ldots, x_n = c_n.$$

Therefore a solution of the system can be considered as an n-tuple vector of $V_n(F)$. In particular, if the field of scalars be $\mathbb{R}$, a solution of the system is a vector in $\mathbb{R}^n$.

A system of equations is said to be *consistent* if it has a solution. Otherwise, it is said to be *inconsistent*.

Examples.

1. $x_1 + 2x_2 = 3$
$3x_1 + x_2 = 4.$

(1,1) is a solution. There is no other solution of the system.

2. $x_1 + 2x_2 = 3$
$3x_1 + 6x_2 = 7.$

This system has no solution. This is not a consistent system.

3. $x_1 + 2x_1 - x_3 = 0$
$2x_1 + x_2 - 2x_3 = 0.$

(1, 0, 1) is a solution of the system. (2, 0, 2), (3, 0, 3) are also solutions. In fact $k(1,0,1)$ is a solution for each real number k. Thus the system has many solutions.

Matrix representation.

Let $A = (a_{ij})_{m,n}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ b_m \end{pmatrix}$.

Then the system (i) can be expressed as $AX = B$. The matrix A is said to be the *coefficient matrix* of the system and the matrix

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$, also denoted by (A, b), is said to be the *augmented matrix* of the system.

The system $AX = B$ is said to be a *homogeneous system* if $B = O$; otherwise, a *non-homogeneous* system.

Two systems $AX = B$ and $CX = D$ are said to be *equivalent systems* if the augmented matrices (A, b) and (C, d) be row equivalent.

Theorem 4.10.1. Let $AX = B$ and $RX = S$ be two equivalent systems and α be a solution of $AX = B$. Then α is also a solution of $RX = S$.

Proof. Let the equations of the system $AX = B$ be

$$\begin{aligned} f_1 &\equiv a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 = 0 \\ f_2 &\equiv a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 = 0 \\ &\cdots \quad \cdots \\ f_m &\equiv a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m = 0 \end{aligned}$$

and let $\alpha = (c_1, c_2, \ldots, c_n)$ be a solution of the system.

Let us apply elementary row operation R_{ij} on the augmented matrix (A, b) of the system. Then the ith and the jth equations f_i and f_j, are interchanged and the others remain unchanged. Therefore $(c_1, c_2, \cdots, c_n)$ is also a solution of the new system.

This implies that if $R_{ij}(A, b) = (C, d)$, then $(c_1, c_2, \cdots, c_n)$ is also a solution of the system $CX = D$.

Let us apply elementary row operation cR_i on the augmented matrix (A, b) of the system. Then the ith equation f_i, is multiplied by $c(\neq 0)$ and the other equations remain unchanged. Therefore $(c_1, c_2, \ldots, c_n)$ is also a solution of the new system.

This implies that if $cR_i(A, b) = (C, d)$, then $(c_1, c_2, \ldots, c_n)$ is also a solution of the system $CX = D$.

Let us apply elementary row operation $R_i + cR_j$ on the augmented matrix (A, b) of the system. Then the ith equation f_i is replaced by $f_i + cf_j$ and the other equations remain unchanged. Therefore $(c_1, c_2, \ldots, c_n)$ is also a solution of the new system.

This implies that if $[R_i + cR_j](A, b) = (C, d)$, then $(c_1, c_2, \ldots, c_n)$ is also a solution of the system $CX = D$.

Since the matrix (R, s) can be obtained from the matrix (A, b) by a finite number of elementary row operations of the above types, a solution of the system $AX = B$ is also a solution of the system $RX = S$.

Corollary. If one of the two equivalent systems be inconsistent, the other is also so.

To examine the solvability of the system $AX = B$, or to determine the solutions (or solution) of the system, when it is consistent, the obvious procedure is to apply such elementary row operations on the augmented matrix (A, b) as will reduce it to a row reduced echelon matrix.

Worked Examples.

1. Solve the system of equations

$$\begin{aligned} x_1 + x_2 &= 4 \\ x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 7. \end{aligned}$$

This is a non-homogeneous system. The coefficient matrix of the system is $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{pmatrix}$ and the augmented matrix is $\bar{A} = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 4 & 7 \end{pmatrix}$.

Let us apply elementary row operations on $\bar{A}$ to reduce it to a row reduced echelon matrix.

$$\bar{A} \xrightarrow{R_3 - 2R_1} \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 4 & -1 \end{pmatrix} \xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3}\begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow[R_2+R_3]{R_1-R_3} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence the system is equivalent to
$$\begin{aligned} x_1 &= 3 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

and therefore the solution is (3, 1, 0).

Note. The coefficient matrix A is row equivalent to the identity matrix I_3 and so it is non-singular. This also suggests that the system admits of a unique solution.

2. Solve the system of equations

$$\begin{aligned} x_1 + 3x_2 + x_2 &= 0 \\ 2x_1 - x_2 + x_3 &= 0. \end{aligned}$$

This is a non-homogeneous system. The coefficient matrix of the system is $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix}$ and the augmented matrix is $\bar{A} = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 2 & -1 & 1 & 0 \end{pmatrix}$.

Let us apply elementary row operations on $\bar{A}$ to reduce it to a row reduced echelon matrix.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 2 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{R_2-2R_1} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -7 & -1 & 0 \end{pmatrix}.$$

$$\xrightarrow{-\frac{1}{7}R_2} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{7} & 0 \end{pmatrix} \xrightarrow{R_1-3R_2} \begin{pmatrix} 1 & 0 & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & 0 \end{pmatrix}.$$

The given system is equivalent to

$$\begin{aligned} x_1 + \tfrac{4}{7}x_3 &= 0 \\ x_2 + \tfrac{1}{7}x_3 &= 0. \end{aligned}$$

Assigning to x_3 an arbitrary real number c, we have the solution $x_1 = -\frac{4}{7}c$, $x_2 = -\frac{1}{7}c$, $x_3 = c$.

Therefore the solution is $\left(-\frac{4}{7}c, -\frac{1}{7}c, c\right)$, i.e., $c\left(-\frac{4}{7}, -\frac{1}{7}, 1\right)$, where c is an arbitrary real number. The solution can be also equivalently expressed as $k(4, 1, -7)$, where k is an arbitrary real number.

Note 1. Instead of considering the augmented matrix we can consider only the coefficient matrix A in the case of a homogeneous system $AX = 0$, since the last column of the augmented matrix is the zero column and this column remains unchanged under elementary row operations.

Note 2. Since k is an arbitrary real number, the number of solutions of the system is infinite. $\alpha = (4, 1, -7)$ is a solution and all solutions are of