Injective mapping. Surjective mapping. Bijective mapping.

A mapping $f : A \to B$ is said to be *injective* (or *one-to-one*) if for each pair of distinct elements of A, their f-images are distinct.

A mapping $f : A \to B$ is said to be *surjective* (or *onto*) if $f(A) = B$.

A mapping $f : A \to B$ is said to be *bijective* if f is both injective and surjective.

An injective mapping is called an *injection*, a surjective mapping is called a *surjection* and a bijective mapping is called a *bijection*.

Thus $f : A \to B$ is injective if $x_1 \neq x_2$ in A implies $f(x_1) \neq f(x_2)$ in B. In this case, each element of B has *at most* one pre-image.

If f is surjective, each element of B has *at least* one pre-image.

If f is bijective, each element of B has *exactly one* pre-image.

In Example 3, f is an injective mapping since for two distinct elements x_1, x_2 in the domain of f, $f(x_1) \neq f(x_2)$. f is not surjective because $f(\mathbb{Z})$ is a proper subset of the co-domain set $\mathbb{Z}$.

In Example 4, f is not injective because two distinct elements 1 and -1 in the domain of f have the same f-image. f is not surjective because $f(\mathbb{Z})$ is a proper subset of the co-domain set $\mathbb{Z}$.

In Example 5, f is injective since for two distinct elements x_1, x_2 in the domain of f, $f(x_1) \neq f(x_2)$. f is surjective because $f(\mathbb{Z})$ = the co-domain set $\mathbb{Z}$. Therefore f is a bijection.

When f is a bijection from A to B, f sets up *a one-to-one correspondence* between the elements of A and the elements of B. Each element x of A is put in correspondence with the single element $f(x)$ of B and each element y of B is put in correspondence with the single element $f^{-1}(y)$ of A.

Definition. A mapping $f : A \to B$ is said to be a *constant mapping* (a constant function) if f maps each element of A to one and the same element of B, i.e., $f(A)$ is a singleton set.

For example, the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2$, $x \in \mathbb{R}$ is a constant mapping. Here $f(\mathbb{R}) = \{2\}$.

Definition. A mapping $f : A \to A$ is said to be the *identity mapping on* A if $f(x) = x, x \in A$. The identity mapping on A is denoted by i_A.

Note. The identity mapping on A is clearly a bijection on A.

Definition. Equality of mappings. Two mappings $f : A \to B$ and $g : A \to C$ are said to be *equal* if $f(x) = g(x)$ for all $x \in A$. For the equality of two mappings f and g, the following conditions must hold:

(i) f and g have the same domain D; and

(ii) for all $x \in D$, $f(x) = g(x)$.

Examples (continued).

9. Let $S = \{x \in \mathbb{R} : x > 0\}$. Let $f : S \to \mathbb{R}$ be defined by $f(x) = \frac{|x|}{x}$, $x \in S$ and $g : S \to \mathbb{R}$ be defined by $g(x) = 1$, $x \in S$. Then $f = g$.

10. Let $S = \{x \in \mathbb{R} : 1 \leq x < 2\}$. Let $f : S \to \mathbb{R}$ be defined by $f(x) = [x] - x$, $x \in S$ and $g : S \to \mathbb{R}$ be defined by $g(x) = 1 - x$, $x \in S$. Then $f = g$.

11. Let $S = \{x \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{2}\}$. Let $f : S \to \mathbb{R}$ be defined by $f(x) = \cos x - \sin x$, $x \in S$ and $g : S \to \mathbb{R}$ be defined by $g(x) = \sqrt{1 - \sin 2x}$, $x \in S$.

Here f and g have the same domain. $f(x) = g(x)$ for $x \in [0, \frac{\pi}{4}]$. But $f(x) \neq g(x)$ for $x \in (\frac{\pi}{4}, \frac{\pi}{2}]$. Therefore $f \neq g$.

Restriction of a mapping.

Let $f : A \to B$ be a mapping and let D be a non-empty subset of A. Then the mapping $g : D \to B$ defined by $g(x) = f(x), x \in D$ is said to be the *restriction* of f to D. This is denoted by f/D.

In this case, f is said to be an *extension* of g to A. An extension $f : A \to B$ of the mapping $g : D \to B$ is not unique, since the f-images of the elements of $A - D$ may be arbitrarily chosen.

When a non-bijective mapping f is given, sometimes it is possible to have a bijective restriction of f.

Examples (continued).

12. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1$, if x be rational
$= 0$, if x be irrational.

Then the restriction mapping $f/\mathbb{Q} : \mathbb{Q} \to \mathbb{R}$ is given by $f/\mathbb{Q}(x) = 1$ for all $x \in \mathbb{Q}$.

The restriction mapping $f/(\mathbb{R} - \mathbb{Q}) : (\mathbb{R} - \mathbb{Q}) \to \mathbb{R}$ is given by $f/(\mathbb{R} - \mathbb{Q})(x) = 0$ for all $x \in (\mathbb{R} - \mathbb{Q})$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $x \in \mathbb{R}$.

Let $S = \mathbb{R}^+$(the set of all positive real numbers). The restriction mapping $f/S : S \to \mathbb{R}$ is defined by $f/S(x) = x^2$, $x \in S$.

Here f is not injective, but the restriction mapping f/S is injective

14. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin\ x$, $x \in \mathbb{R}$. f is neither surjective nor injective.

If we reduce the co-domain to $T = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$, then the mapping $f : \mathbb{R} \to T$ defined by $f(x) = \sin\ x$, $x \in \mathbb{R}$ is surjective, but not injective.

Let $S = \{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$. Then the restriction mapping $g : S \to T$ defined by $g(x) = \sin\ x$, $x \in S$ is a bijection.

Worked Examples.

1. A mapping $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x + 1, x \in \mathbb{R}$. Examine if (i) f is injective, (ii) f is surjective.

(i) Let us take two distinct elements x_1, x_2 in $\mathbb{R}$, the domain of f.

$f(x_1) = 3x_1 + 1, f(x_2) = 3x_2 + 1$.

$f(x_1) - f(x_2) = 3(x_1 - x_2) \neq 0$, since $x_1 \neq x_2$.

Since $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, f is injective.

(ii) Let us take an arbitrary element y in the set $\mathbb{R}$, the co-domain of f; and let us examine if y has a pre-image x in the domain of f.

Then $f(x) = y$ and therefore $3x + 1 = y$ or, $x = \frac{y-1}{3}$.

Since $y \in \mathbb{R}$, $\frac{y-1}{3} \in \mathbb{R}$. Therefore y has a pre-image $\frac{y-1}{3}$ in the domain of f. Since y is arbitrary, each element in the co-domain of f has a pre-image under f. Therefore f is surjective.

2. A mapping $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, $x \in \mathbb{R}$. Examine if (i) f is injective, (ii) f is surjective.

(i) $f(2) = 4, f(-2) = 4$. f is not injective, since two distinct elements 2 and -2 in the domain of f have the same image.

(ii) f is not surjective, since -1 in the co-domain of f has no pre-image in the domain of f.

1.11. Composition of mappings.

Let $f : A \to B$ and $g : C \to D$ be two mappings such that $f(A)$ is a subset of C.

Let $x \in A$. Then f maps x to an element y in $f(A) \subset B$ and since $y \in f(A) \subset C$, g maps y to an element z in D.

We can conceive of a mapping $h : A \to D$ defined by $h(x) = g(f(x))$, $x \in A$. The mapping $h : A \to D$ is said to be the *composite* (or the *product*) of f and g and is denoted by $g \circ f$ (or by gf). The composite $g \circ f : A \to D$ is defined only if $f(A)$ is a subset of the domain of g.

For the mappings $f : A \to B$ and $g : B \to C$, the composite g $A \to C$ is defined. For the mappings $f : A \to B$ and $g : B \to A$ both composites $g \circ f : A \to A$ and $f \circ g : B \to B$ are defined.

Examples.

1. Let $A = \{1, 2, 3\}, B = \{p, q, r\}$, and let
$f : A \to B$ be defined by $f(1) = p, f(2) = q, f(3) = r$;
$g : B \to A$ be defined by $g(p) = 3, g(q) = 2, g(r) = 1$;
$h : B \to A$ be defined by $h(p) = 1, h(q) = 2, h(r) = 3$.

$g \circ f : A \to A$ is defined by $g \circ f(1) = 3, g \circ f(2) = 2, g \circ f(3) = 1$.

$f \circ g : B \to B$ is defined by $f \circ g(p) = r, f \circ g(q) = q, f \circ g(r) = p$.

$h \circ f : A \to A$ is defined by $h \circ f(1) = 1, h \circ f(2) = 2, h \circ f(3) = 3$.

$f \circ h : B \to B$ is defined by $f \circ h(p) = p, f \circ h(q) = q, f \circ h(r) = r$.

Here $g \circ f \neq f \circ g$, $h \circ f = i_A$, $f \circ h = i_B$.

2. Let $f : \mathbb{Z} \to \mathbb{Q}$ be defined by $f(x) = \frac{1}{2}x$, $x \in \mathbb{Z}$ and $g : \mathbb{Q} \to \mathbb{Q}$ b defined by $g(x) = x^2$, $x \in \mathbb{Q}$.

$g \circ f : \mathbb{Z} \to \mathbb{Q}$ is defined by $g \circ f(x) = g(\frac{1}{2}x) = \frac{1}{4}x^2$, $x \in \mathbb{Z}$.

For example, $g \circ f(1) = g(f(1)) = g(\frac{1}{2}) = \frac{1}{4}$. $g \circ f(2) = g(f(2)) =$ $g(1) = 1$. Here $f \circ g$ is not defined, since the range of g is not a subset of the domain of f.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$, $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 3x$, $x \in \mathbb{R}$.

Here $g \circ f$ and $f \circ g$ are both defined.

$g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by $g \circ f(x) = g(x + 1) = 3x + 3$, $x \in \mathbb{R}$;
$f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by $f \circ g(x) = f(3x) = 3x + 1$, $x \in \mathbb{R}$.

Both the mappings $g \circ f$ and $f \circ g$ have the same domain but $g \circ f(x) \neq$ $f \circ g(x), x \in \mathbb{R}$. Therefore $g \circ f \neq f \circ g$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$, $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x + 5$, $x \in \mathbb{R}$.

Here both $g \circ f$ and $f \circ g$ are defined.

$f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by $f \circ g(x) = f(x + 5) = x + 6$, $x \in \mathbb{R}$;
$g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by $g \circ f(x) = g(x + 1) = x + 6$, $x \in \mathbb{R}$.

Since $g \circ f(x) = f \circ g(x)$ for all $x \in \mathbb{R}$, we have $g \circ f = f \circ g$.

It is quite clear from the examples that the composition of mappings is not commutative. That is, for two mappings f and g their composites

$g \circ f$ and $f \circ g$ are, in general, not equal. In fact, when one of them is defined, the other may not be defined at all. In particular cases, however, the equality holds.

Although the composition of mappings is not commutative, it is associative. That is, for three mappings f, g and h, $h \circ (g \circ f) = (h \circ g) \circ f$, when both sides are defined mappings.

Theorem 1.11.1. Let $f : A \to B$, $g : B \to C$, $h : C \to D$ be three mappings. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. Here the composite mappings $g \circ f, h \circ g$ are defined because the range $f \subset$ dom g and the range $g \subset$ dom h. The composite mappings $h \circ (g \circ f), (h \circ g) \circ f$ are defined because the range $g \circ f \subset$ dom h and the range $f \subset$ dom $h \circ g$.

We now prove the equality of the mappings $h \circ (g \circ f) : A \to D$ and $(h \circ g) \circ f : A \to D$.

Let x be an element of A and let $f(x) = y$, $g(y) = z$, $h(z) = w$.
Then $g \circ f(x) = g(y) = z$, $\quad h \circ g(y) = h(z) = w$.
$h \circ (g \circ f) : A \to D$ is defined by $h \circ (g \circ f)(x) = h(z) = w$, $x \in A$.
$(h \circ g) \circ f : A \to D$ is defined by $(h \circ g) f(x) = h \circ g(y) = w$, $x \in A$.

Since $h \circ (g \circ f)(x) = (h \circ g) \circ f(x)$ for all $x \in A$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

Theorem 1.11.2. If $f : A \to B$ and $g : B \to C$ be both injective mappings, then the composite mapping $g \circ f : A \to C$ is injective.

Proof. Let x_1, x_2 be two distinct elements of A.
Let $f(x_1) = y_1$, $f(x_2) = y_2$.
Since f is injective, y_1 and y_2 are distinct elements of B.
Let $g(y_1) = z_1$, $g(y_2) = z_2$.
Since g is injective, z_1 and z_2 are distinct elements of C.
Now $g \circ f(x_1) = g(y_1) = z_1$, $g \circ f(x_2) = g(y_2) = z_2$ and $x_1 \neq x_2$ in A $\Rightarrow z_1 \neq z_2$ in C. Therefore $g \circ f$ is injective.

Note. The converse of the theorem is not true. However if $g \circ f$ is injective, then f is injective (while g need not be).

Theorem 1.11.3. If $f : A \to B$ and $g : B \to C$ be two mappings such that $g \circ f : A \to C$ is injective, then f is injective.

Proof. Let $f(x_1) = f(x_2)$ for some x_1, x_2 in A. Then $g[f(x_1)] = g[f(x_2)]$, since g is a mapping. So $g \circ f(x_1) = g \circ f(x_2)$.
Because $g \circ f$ is injective, $g \circ f(x_1) = g \circ f(x_2)$ implies $x_1 = x_2$.
Thus $f(x_1) = f(x_2)$ implies $x_1 = x_2$ and therefore f is injective.

Note. In order that $g \circ f$ may be injective, it is not necessary that g injective.

For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$, $x \in \mathbb{R}$ a
$g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$, $x \in \mathbb{R}$.

Here $g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by $g \circ f(x) = e^{2x}$, $x \in \mathbb{R}$.

$g \circ f$ is injective but g is not injective.

Theorem 1.11.4. If $f : A \to B$ and $g : B \to C$ be both surjective, th the composite mapping $g \circ f : A \to C$ is surjective.

Proof. Let z be an element of C.

Since g is surjective, there is at least one pre-image of z in B. L one such be y. Then $y \in B$ and $g(y) = z$.

Since f is surjective and $y \in B$, there is at least one pre-image of in A. Let one such be x. Then $x \in A$ and $f(x) = y$.

$g \circ f(x) = g(y) = z$. This implies that z has a pre-image x in A und the mapping $g \circ f$. Since z is arbitrary, $g \circ f$ is surjective.

Note. The converse of the theorem is not true. However if $g \circ f$ is su jective, then g is surjective (while f need not be).

Theorem 1.11.5. If $f : A \to B$ and $g : B \to C$ be two mappings su that $g \circ f : A \to C$ is surjective, then g is surjective.

Proof. Let z be an element of C. Since $g \circ f$ is surjective, there is element x in A such that $g \circ f(x) = z$. Therefore $g(f(x)) = z$.

This shows that z has a pre-image $f(x)$ in B under the mapping Since z is arbitrary, g is surjective.

Note. In order that $g \circ f$ may be surjective, it is not necessary that f surjective.

For example, let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$ a $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = [\frac{x}{2}]$, $x \in \mathbb{Z}$. $[x]$ denotes the greate integer$\leq x$. Then $g \circ f : \mathbb{Z} \to \mathbb{Z}$ is defined by $g \circ f(x) = x, x \in \mathbb{Z}$.

$g \circ f = i_{\mathbb{Z}}$ and therefore $g \circ f$ is surjective but f is not surjective.

Theorem 1.11.6. If $f : A \to B$ and $g : B \to C$ be both bijective, th the composite mapping $g \circ f : A \to C$ is bijective.

This is a combination of the Theorems 1.11.2 and 1.11.4.

Note. The converse of the theorem is not true. However if $g \circ f$ is bij tive, then f is injective and g is surjective.

This follows from the Theorems 1.11.3 and 1.11.5.