1.12. Inverse mapping.

Definition. Let $f : A \to B$ be a mapping. If there exists a mapping $g : B \to A$ such that $g \circ f = i_A$, then g is said to be a *left inverse* of f. If there exists a mapping $h : B \to A$ such that $f \circ h = i_B$, then h is said to be a *right inverse* of f.

Definition. Let $f : A \to B$ be a mapping. f is said to be *invertible* if there exists a mapping $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$. In this case g is said to be *an inverse* of f.

Theorem 1.12.1. If $f : A \to B$ be an invertible mapping, then its inverse is unique.

Proof. Since $f : A \to B$ is invertible, there exists a mapping $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.

If possible, let there exist another mapping $h : B \to A$ such that $h \circ f = i_A$ and $f \circ h = i_B$.

$h \circ (f \circ g) = (h \circ f) \circ g$, since composition of mappings is associative.
Therefore $h \circ i_B = i_A \circ g$, i.e., $h = g$. This proves that g is unique.

Note 1. g is said to be *the* inverse of f and is denoted by f^{-1}. Therefore $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

2. If $f : A \to A$ be an invertible mapping, then $f^{-1} \circ f = f \circ f^{-1} = i$, where i is the identity mapping on A.

Examples.

1. Let $f : \mathbb{R} \to \mathbb{Z}$ be defined by $f(x) = [x]$, $x \in \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{R}$ be defined by $g(x) = x + \frac{1}{2}$, $x \in \mathbb{Z}$.

$f \circ g : \mathbb{Z} \to \mathbb{Z}$ is defined by $f \circ g(x) = f(x + \frac{1}{2}) = [x + \frac{1}{2}] = x$, $x \in \mathbb{Z}$.
$g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by $g \circ f(x) = g([x]) = [x] + \frac{1}{2}$, $x \in \mathbb{R}$.
Here $f \circ g = i_{\mathbb{Z}}$, $g \circ f \neq i_{\mathbb{R}}$.
Therefore g is a right inverse of f, but not a left inverse of f.

Let $h : \mathbb{Z} \to \mathbb{R}$ be defined by $h(x) = x + \frac{1}{3}$, $x \in \mathbb{Z}$.
Then $f \circ h = i_{\mathbb{Z}}$ and therefore h is a right inverse of f.

Note. There are many right inverses of f. Since f is not a bijection, f is not invertible.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x$, $x \in \mathbb{R}$ and $g : \mathbb{R}$ ...
defined by $g(x) = \frac{x}{3}$, $x \in \mathbb{R}$.

$gof : \mathbb{R} \to \mathbb{R}$ is defined by $gof(x) = g(3x) = x$, $x \in \mathbb{R}$

$fog : \mathbb{R} \to \mathbb{R}$ is defined by $fog(x) = f(\frac{x}{3}) = x$, $x \in \mathbb{R}$.

Here $gof = fog = i_R$. Threfore g is the inverse of f.

Theorem 1.12.2. A mapping $f : A \to B$ is invertible if and only a bijection.

Proof. Let $f : A \to B$ be invertible. Then there exists a ma $g : B \to A$ such that $gof = i_A$ and $fog = i_B$.

Since i_A is injective and $gof = i_A$, f is an injection.

Since i_B is surjective and $fog = i_B$, f is a surjection.

Therefore f is a bijection.

Conversely, let $f : A \to B$ be a bijection.

Let $y \in B$. Since f is a bijection, y has a unique pre-image x in A.

Define a mapping $g : B \to A$ by $g(y) = x$ (the pre-image of y u f), $y \in B$. Then $gof(x) = g(y) = x$, $x \in A$ and $fog(y) = f(x) = y$, B.

Here $gof = i_A$, $fog = i_B$ and therefore f is invertible.

Note. The theorem gives a clue how to determine f^{-1}. To each elem y in B, f^{-1} assigns the pre-image of y under f.

Theorem 1.12.3. Let $f : A \to B$ be a bijective mapping. Then t mapping $f^{-1} : B \to A$ is also a bijection and $(f^{-1})^{-1} = f$.

Proof. Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists and $f^{-1}of$ i_A and $fof^{-1} = i_B$.

Let y_1, y_2 be two distinct elements in B and $f^{-1}(y_1) = x_1, f^{-1}(y_2) = x$

Then $f(x_1) = f[f^{-1}(y_1)] = fof^{-1}(y_1) = y_1$ and $f(x_2)$ $f[f^{-1}(y_2)] = fof^{-1}(y_2) = y_2$, since $fof^{-1} = i_B$.

Since f is a mapping, $y_1 \neq y_2 \Rightarrow x_1 \neq x_2$.

That is, $y_1 \neq y_2 \Rightarrow f^{-1}(y_1) \neq f^{-1}(y_2)$ and therefore f^{-1} is injecti

To establish that f^{-1} is surjective, let $x \in A$. Let $f(x) = y$.

Then $f^{-1}(y) = f^{-1}[f(x)] = f^{-1}of(x) = x$, since $f^{-1}of = i_A$.

This shows that y is a pre-image of x under the mapping f^{-1}. There fore f^{-1} is surjective.

Since f^{-1} is injective as well as surjective, f^{-1} is a bijection.

Second part.

Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists and $f^{-1}\circ f = i_A$ and $f\circ f^{-1} = i_B$. Since $f^{-1} : B \to A$ is a bijection, $(f^{-1})^{-1} : A \to B$ exists and $f^{-1}\circ(f^{-1})^{-1} = i_A$ and $(f^{-1})^{-1}\circ f^{-1} = i_B$.

Let $x \in A$ and $f(x) = y$.

$f^{-1}(y) = f^{-1}[f(x)] = f^{-1}\circ f(x) = x$, since $f^{-1}\circ f = i_A$.

$(f^{-1})^{-1}(x) = (f^{-1})^{-1}[f^{-1}(y)] = (f^{-1})^{-1}\circ f^{-1}(y) = y$, since $(f^{-1})^{-1}\circ f^{-1} = i_B$.

Therefore $f(x) = (f^{-1})^{-1}(x)$ for all $x \in A$.

Consequently, $f = (f^{-1})^{-1}$. This completes the proof.

Theorem 1.12.4. Let $f : A \to B$ and $g : B \to C$ be both bijective mappings. Then the mapping $g\circ f : A \to C$ is invertible and $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.

Proof. Since $f : A \to B$ and $g : B \to C$ are both bijective, the composite mapping $g\circ f : A \to C$ is a bijection, by Theorem 1.11.6.

Hence $(g\circ f)^{-1} : C \to A$ exists, by Theorem 1.12.2.

Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists.
Since $g : B \to C$ is a bijection, $g^{-1} : C \to B$ exists.
Then the composite mapping $f^{-1}\circ g^{-1} : C \to A$ exists.

Let $z \in C$. Let y be the pre-image of z under the bijective mapping g and x be the pre-image of y under the bijective mapping f.

Then $g\circ f(x) = z$. Since $g\circ f$ is invertible, $(g\circ f)^{-1}(z) = x$.

Since f is invertible and $f(x) = y$, $f^{-1}(y) = x$.
Since g is invertible and $g(y) = z$, $g^{-1}(z) = y$.
Therefore $f^{-1}\circ g^{-1}(z) = f^{-1}[g^{-1}(z)] = f^{-1}(y) = x$.
Thus $(g\circ f)^{-1}(z) = (f^{-1}\circ g^{-1})(z)$ for all $z \in C$.

Consequently, $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$. This completes the proof.

Worked Examples.

1. A and B are both finite sets of n elements. Prove that a mapping $f : A \to B$ is injective if and only if it is surjectve.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and let $f : A \to B$ be an injective mapping. Then the f-images $f(a_1), f(a_2), \ldots, f(a_n)$ are n distinct elements of B. Since $|B| = n$, these images cover the whole of B. Therefore f is surjective.

Conversely, let f be surjective. Then the images $f(a_1), f(a_2), \ldots, f(a_n)$ cover the whole of B. Since $|B| = n$, $f(a_1), f(a_2), \ldots, f(a_n)$ are n distinct elements of B and therefore these are n distinct images of the elements of A. So f is injective.

2. A mapping $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x + 1$, $x \in \mathbb{R}$. Prove that f is invertible. Find f^{-1}.

As f is injective and surjective (worked Ex.1, 1.10), f is a bijection and therefore f is invertible.

Each y in the co-domain of f has a unique pre-image $\frac{y-1}{3}$.

$f^{-1} : \mathbb{R} \to \mathbb{R}$ is defined by $f^{-1}(y) = \frac{y-1}{3}$, $y \in \mathbb{R}$; or equivalently defined by $f^{-1}(x) = \frac{x-1}{3}$, $x \in \mathbb{R}$.

3. Let $S = \{x \in \mathbb{R} : -1 < x < 1\}$. A map $f : \mathbb{R} \to S$ is defined by $f(x) = \frac{x}{1+|x|}$, $x \in \mathbb{R}$. Show that f is a bijection. Determine f^{-1}.

Let $x > 0$. Then $f(x) = \frac{x}{1+x}$ and $0 < f(x) < 1$.
Let $x = 0$. Then $f(x) = 0$.
Let $x < 0$. Then $f(x) = \frac{x}{1-x} = \frac{1}{1-x} - 1$ and $-1 < f(x) < 0$.
Let x_1, x_2 belong to $\mathbb{R}$ and $f(x_1) = f(x_2)$. Then $\frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}$.
This implies x_1 and x_2 are either both positive, or both negative.
Let $x_1 > 0$ and $x_2 > 0$.
Then $f(x_1) = f(x_2) \Rightarrow \frac{x_1}{1+x_1} = \frac{x_2}{1+x_2} \Rightarrow x_1 = x_2$.
Let $x_1 < 0$ and $x_2 < 0$.
Then $f(x_1) = f(x_2) \Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1 = x_2$.
It follows that $x_1 \neq x_2$ in $\mathbb{R} \Rightarrow f(x_1) \neq f(x_2)$. So f is injective.

Let y be an element in S and let $0 < y < 1$. Let us examine if $x(> 0)$ in $\mathbb{R}$ be a pre-image of y. Then $\frac{x}{1+x} = y$
or, $x = \frac{y}{1-y} \in \mathbb{R}$, since $y \in \mathbb{R}$. Therefore $\frac{y}{1-y}$ is a pre-image of y.
Let y be an element in S and let $-1 < y < 0$. Let us examine if $x(< 0)$ in $\mathbb{R}$ be a pre-image of y. Then $\frac{x}{1-x} = y$
or, $x = \frac{y}{1+y} \in \mathbb{R}$, since $y \in \mathbb{R}$. Therefore $\frac{y}{1+y}$ is a pre-image of y.
Let $y = 0 \in S$. Then $x = 0$ is a pre-image of y.
It follows that each y in S has a pre-image in $\mathbb{R}$. So f is surjective.
Since f is injective as well as surjective, f is a bijection.
Since f is a bijection, each y in S has a unique pre-image.
For $y > 0$, the pre-image is $\frac{y}{1-y} = \frac{y}{1-|y|}$.
For $y < 0$, the pre-image is $\frac{y}{1+y} = \frac{y}{1-|y|}$.
For $y = 0$, the pre-image is $0 = \frac{y}{1-|y|}$.
It follows that for each y in S the pre-image is $\frac{y}{1-|y|}$.
Hence $f^{-1} : S \to \mathbb{R}$ is defined by $f^{-1}(x) = \frac{x}{1-|x|}$, $x \in S$.