7.3. Difference operator Δ.

Let us consider a sequence $(u_1, u_2, \ldots u_n, \ldots \ldots)$ where u_n is a function of positive integral variable n.

We define $\Delta u_n = u_{n+1} - u_n$.
Then $\Delta u_1 = u_2 - u_1, \Delta u_2 = u_3 - u_2, \Delta u_3 = u_4 - u_3, \cdots \cdots$

The sequence $(\Delta u_1, \Delta u_2, \ldots \Delta u_n, \ldots \ldots)$ is said to be the *first order of differences* of the sequence (u_n).

We define $\Delta^2 u_n = \Delta(\Delta u_n) = \Delta(u_{n+1} - u_n)$.

The sequence $(\Delta^2 u_1, \Delta^2 u_2, \Delta^2 u_3, \ldots \ldots)$ is said to be the *second order of differences* of the sequence (u_n).

In a similar manner, the pth order of differences $\Delta^p u_n$ where p is a positive integer, can be defined for the sequence (u_n).

It follows immediately that

(i) if $u_n = k$, a constant for all n, then $\Delta u_n = 0$ for all n, and conversely;

(ii) If (u_n) be a sequence in an arithmetic progression having the common difference d, then Δu_n = constant $= d$ for all n and conversely.

(iii) If (u_n) be a sequence in a geometric progression having the common ratio r, then the sequence (Δu_n) is also in geometric progression of common ratio r.

Examples.

1. Let the sequence be $(8, 18, 34, 56, 84, 118, \ldots\ \ldots)$.

The successive orders of differences are

$$
\begin{array}{ccccccccccc}
10 & & 16 & & 22 & & 28 & & 34 & \ldots & \\
& 6 & & 6 & & 6 & & 6 & \ldots & & \\
& & 0 & & 0 & & 0 & \ldots & \ldots & \ldots &
\end{array}
$$

Here the second order of differences is a sequence of equal elements. The third and the successive orders of differences are sequences of zeros.

2. Let the sequence be $(4, 12, 36, 108, 324, 972, \ldots\ \ldots)$.

The successive orders of differences are

$$
\begin{array}{cccccccccc}
8 & & 24 & & 72 & & 216 & & 648 & \ldots \\
& 16 & & 48 & & 144 & & 432 & \ldots & \ldots
\end{array}
$$

Here the given sequence is in a geometric progression of common ratio 3. The successive orders of differences are also in geometric progression of common ratio 3.

Let us consider the sequence $(u_1, u_2, u_3, \cdots\ \cdots)$ and the successive orders of differences.

$$
\begin{array}{cccccccccccc}
u_1 & & u_2 & & u_3 & & u_4 & & u_5 & \ldots & & \\
& \Delta u_1 & & \Delta u_2 & & \Delta u_3 & & \Delta u_4 & \ldots & \ldots & & \\
& & \Delta^2 u_1 & & \Delta^2 u_2 & & \Delta^2 u_3 & \ldots & \ldots & \ldots & & \\
& & & \Delta^3 u_1 & & \Delta^3 u_2 & \ldots & \ldots & \ldots & \ldots & & \\
& & & \ldots & & \ldots & & \ldots & & & &
\end{array}
$$

It appears from the table that

$$\begin{aligned}
u_2 &= u_1 + \Delta u_1, \\
u_3 &= u_2 + \Delta u_2 \\
&= (u_1 + \Delta u_1) + \Delta(u_1 + \Delta u_1) \\
&= u_1 + 2\Delta u_1 + \Delta^2 u_1,
\end{aligned}$$

$$\begin{aligned}
u_4 &= u_3 + \Delta u_3 \\
&= (u_2 + \Delta u_2) + \Delta(u_2 + \Delta u_2) \\
&= u_2 + 2\Delta u_2 + \Delta^2 u_2 \\
&= (u_1 + \Delta u_1) + 2\Delta(u_1 + \Delta u_1) + \Delta^2(u_1 + \Delta u_1) \\
&= u_1 + 3\Delta u_1 + 3\Delta^2 u_1 + \Delta^3 u_1, \\
&\quad \cdots \quad \cdots \quad \cdots
\end{aligned}$$

Theorem 7.3.1. Let (u_n) be a sequence and Δ^p denote the pth order difference operator. Then

$$u_n = u_1 + \binom{n-1}{1}\Delta u_1 + \binom{n-1}{2}\Delta^2 u_1 + \cdots + \binom{n-1}{r}\Delta^r u_1 + \cdots + \Delta^{n-1} u_1.$$

Proof. We prove the theorem by the method of induction.

The theorem is obviously true for $n = 1$.

Let us assume the theorem to be true for $n = m$, where m is a positive integer.

Then $u_m = u_1 + \binom{m-1}{1}\Delta u_1 + \binom{m-1}{2}\Delta^2 u_1 + \cdots + \binom{m-1}{r}\Delta^r u_1 + \cdots + \Delta^{m-1} u_1$.

Therefore $\Delta u_m = \Delta u_1 + \binom{m-1}{1}\Delta^2 u_1 + \binom{m-1}{2}\Delta^3 u_1 + \cdots + \binom{m-1}{r}\Delta^{r+1} u_1 + \cdots + \Delta^m u_1$.

$$\begin{aligned}
\text{Hence } u_{m+1} &= u_m + \Delta u_m \\
&= u_1 + \left[1 + \binom{m-1}{1}\right]\Delta u_1 + \left[\binom{m-1}{1} + \binom{m-1}{2}\right]\Delta^2 u_1 + \cdots \\
&\quad + \left[\binom{m-1}{r-1} + \binom{m-1}{r}\right]\Delta^r u_1 + \cdots + \Delta^m u_1 \\
&= u_1 + \binom{m}{1}\Delta u_1 + \cdots + \binom{m}{r}\Delta^r u_1 + \cdots + \Delta^m u_1.
\end{aligned}$$

This shows that the theorem is true for $n = m + 1$ if it be true for $n = m$. And the theorem is true for $n = 1$.

By the principle of induction, the theorem holds for all positive integers n.

Corollary. If p be the least positive integer such that the sequence $(\Delta^p u_n)$ is a sequence of equal elements, then u_n is a polynomial in n of degree p. Because in this case all the sequences of $(p+1)$th and the higher orders of differences are sequences of zeros.

Therefore $u_n = u_1 + (n-1)\Delta u_1 + \cdots + (n-1)(n-2)\cdots(n-p)\frac{\Delta^p u_1}{p!}$

$= \text{a polynomial in } n \text{ of degree } p.$

Worked Examples.

1. Find u_n, where the sequence (u_n) is $(3, 10, 21, 36, 55, \ldots\ \ldots)$.

The successive orders of differences are

$$\begin{array}{ccccccccc} 7 & & 11 & & 15 & & 19 & \ldots & \\ & 4 & & 4 & & 4 & & \ldots & \ldots \end{array}$$

$$\begin{aligned} \text{Therefore } u_n &= 3 + (n-1)7 + \tfrac{(n-1)(n-2)}{2!}.4 \\ &= 2n^2 + n. \end{aligned}$$

Alternatively, u_n is a polynomial in n of degree 2, since the second order of differences is a sequence of equal elements.

Let $u_1 = an^2 + bn + c$.

$$\begin{aligned} \text{Since } u_1 &= 3, & a + b + c &= 3; \\ \text{since } u_2 &= 10, & 4a + 2b + c &= 10; \\ \text{since } u_3 &= 21, & 9a + 3b + c &= 21. \end{aligned}$$

These determine $a = 2, b = 1, c = 0$.

Therefore $u_n = 2n^2 + n$.

2. Find the sum to n terms of the series

$$3 + 11 + 31 + 69 + 131 + \cdots$$

Let the series be $u_1 + u_2 + u_3 + \cdots$

The sequence (u_n) and the successive orders of differences are

$$\begin{array}{cccccccccccc} 3 & & 11 & & 31 & & 69 & & 131 & \ldots & & \\ & 8 & & 20 & & 38 & & 62 & & \ldots & \ldots & \\ & & 12 & & 18 & & 24 & \ldots & & \ldots & \ldots & \\ & & & 6 & & 6 & \ldots & & \ldots & & \ldots & \ldots \end{array}$$

$$\begin{aligned} \text{Therefore } u_n &= 3 + 8(n-1) + 12.\tfrac{(n-1)(n-2)}{2} + 6.\tfrac{(n-1)(n-2)(n-3)}{6} \\ &= n^3 + n + 1. \end{aligned}$$

$$\begin{aligned} \text{Hence } \sum_1^n u_n &= \tfrac{n^2(n+1)^2}{4} + \tfrac{n(n+1)}{2} + n \\ &= n[\tfrac{(n^3+2n^2+n)+(2n+2)+4}{4}] \\ &= n(\tfrac{n^3+2n^2+3n+6}{4}). \end{aligned}$$

3. Find the nth term and the sum to n terms of the series

$$6.3 + 9.8 + 12.15 + 15.24 + 18.35 + \cdots\ \cdots$$

Let $u_1 + u_2 + u_3 + \cdots\ \cdots$ be the given series. Let (v_n) and (w_n) be the sequences $(6, 9, 12, 15, 18, \ldots\ \ldots)$, $(3, 8, 15, 24, 35, \ldots\ \ldots)$ respectively.

The successive orders of differences of the sequence (w_n) are

$$\begin{array}{ccccccccccc}
3 & & 8 & & 15 & & 24 & & 35 & \cdots \\
& 5 & & 7 & & 9 & & 11 & \cdots & \cdots \\
& & 2 & & 2 & & 2 & \cdots & \cdots & \cdots
\end{array}$$

$$\begin{aligned}
\text{Therefore } w_n &= 3 + 5.(n-1) + 2.\tfrac{(n-1)(n-2)}{2} \\
&= n^2 + 2n; \\
\text{and } u_n &= 3n(n+1)(n+2).
\end{aligned}$$

Hence $\sum\limits_1^n u_r = \dfrac{3n(n+1)(n+2)(n+3)}{4} + c$ where c is a constant.

But $u_1 = 18$. $18 = \frac{3.2.3.4}{4} + c$ and this gives $c = 0$.

Therefore $\sum\limits_1^n u_r = \dfrac{3}{4}n(n+1)(n+2)(n+3)$.

4. Find the sum to n terms of the the series

$$2.1 + 7.2 + 14.2^2 + 23.2^3 + 34.2^4 + \cdots \cdots$$

Let (v_n) be the sequence $(1, 7, 14, 23, 34, \cdots \cdots)$.

The successive orders of differences are

$$\begin{array}{cccccccc}
5 & & 7 & & 9 & & 11 & \cdots \\
& 2 & & 2 & & 2 & \cdots & \cdots
\end{array}$$

$$\begin{aligned}
\text{Therefore } v_n &= 2 + (n-1)5 + \tfrac{(n-1)(n-2)}{2}.2 \\
&= n^2 + 2n - 1.
\end{aligned}$$

Let the given series be $u_1 + u_2 + u_3 + \cdots \cdots$
Then $u_n = (n^2 + 2n - 1).2^{n-1}$.

Let $(n^2 + 2n - 1).2^{n-1} = f(n).2^n - f(n-1).2^{n-1}$, where $f(n)$ is a polynomial of degree 2. Let us assume $f(n) = an^2 + bn + c$.

Then $(n^2 + 2n - 1).2^{n-1}$

$$\begin{aligned}
&= (an^2 + bn + c).2^n - \{a(n-1)^2 + b(n-1) + c\}.2^{n-1} \\
&= \{2an^2 + 2bn + 2c - a(n-1)^2 - b(n-1) - c\}.2^{n-1} \\
&= \{an^2 + (b + 2a)n - a + b + c\}.2^{n-1}.
\end{aligned}$$

Equating, we have $a = 1, b = 0, c = 0$.
So $u_n = n^2.2^n - (n-1)^2.2^{n-1}$.

$$\begin{aligned}
\text{Therefore } u_1 &= 1.2 \\
u_2 &= 2^2.2^2 - 1.2 \\
u_3 &= 3^2.2^3 - 2^2.2^2 \\
\cdots & \quad \cdots \quad \cdots \\
u_n &= n^2.2^n - (n-1)^2.2^{n-1}.
\end{aligned}$$

By addition, $\sum\limits_1^n u_r = n^2.2^n$.

Let us consider the sequence (u_n), where $u_n = ar^{n-1}$. Then the sequence is $(a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots \ldots)$, a geometric progression of common ratio r.

The successive orders of differences are

$$\begin{array}{cccccccc}
a(r-1) & & ar(r-1) & & ar^2(r-1) & & \cdots & \cdots \\
& a(r-1)^2 & & ar(r-1)^2 & & ar^2(r-1)^2 & & \cdots \\
& & a(r-1)^3 & & ar(r-1)^3 & & \cdots & \cdots
\end{array}$$

Here each order of differences is a sequence in geometric progression of common ratio r.

Let us consider the sequence (u_n), where $u_n = k + ar^{n-1}, k$ being a constant.

$$\begin{aligned}
\text{Then } \Delta u_n &= u_{n+1} - u_n \\
&= (k + ar^n) - (k + ar^{n-1}) \\
&= ar^{n-1}(r - 1.)
\end{aligned}$$

This shows that the first order of differences is a sequence in geometric progression of common ratio r. Therefore the successive orders of differences are also sequences in geometric progression of common ratio r.

Let $u_n = \phi(n) + ar^{n-1}$, where $\phi(n)$ is a polynomial of degree 1.

$$\begin{aligned}
\text{Then } \Delta u_n &= \phi(n+1) - \phi(n) + ar^{n-1}(r-1) \\
&= k + ar^{n-1}(r-1), k \text{ being a constant.} \\
\Delta^2 u_n &= ar^{n-1}(r-1)^2.
\end{aligned}$$

Here the second and the successive orders of differences are sequences in geometric progression of common ratio r.

Let $u_n = \phi(n) + ar^{n-1}$, where $\phi(n)$ is a polynomial of degree p. Then the $(p+1)$th and the successive orders of differences are sequences in geometric progression of common ratio r.

Worked Examples (continued).

5. Find u_n, where the sequence (u_n) is $(7, 14, 33, 88, 251, \ldots \ldots)$.

The succesive orders of differences are

$$\begin{array}{ccccccc}
7 & & 19 & & 55 & & 163 \quad \cdots \\
& 12 & & 36 & & 108 & \cdots \quad \cdots \\
& & 24 & & 72 & \cdots & \cdots \quad \cdots
\end{array}$$

Here the second and the successive orders of differences are sequences in geometric progression of common ratio 3. Therefore $u_n = a.3^{n-1} + \phi(n)$, where $\phi(n)$ is a polynomial of degree 1.

Let $u_n = a.3^{n-1} + bn + c.$

Then $a + b + c = 7, \quad 3a + 2b + c = 14, \quad 9a + 3b + c = 33.$

These determine $a = 3, b = 1, c = 3.$

Therefore $u_n = 3^n + n + 3.$

6. Find the sum of the first n terms of the series

$$7 + 14 + 33 + 88 + 251 + \cdots \cdots$$

By the previous example, the nth term u_n is given by

$$u_n = 3^n + n + 3.$$

$$\text{Therefore } \sum_1^n u_r = 3(1 + 3 + 3^3 + \cdots + 3^{n-1}) + \tfrac{n(n+1)}{2} + 3n$$

$$= 3.\tfrac{3^n - 1}{2} + \tfrac{n(n+1)}{2} + 3n.$$